

# Exact Localized Solutions of Quintic Discrete Nonlinear Schrödinger Equation

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## Abstract

We study a new quintic discrete nonlinear Schrödinger (QDNLS) equation which reduces naturally to an interesting symmetric difference equation of the form  $\phi_{n+1} + \phi_{n-1} = F(\phi_n)$ . Integrability of the symmetric mapping is checked by singularity confinement criteria and growth properties. Some new exact localized solutions for integrable cases are presented for certain sets of parameters. Although these exact localized solutions represent only a small subset of the large variety of possible solutions admitted by the QDNLS equation, those solutions presented here are the first example of exact localized solutions of the QDNLS equation. We also find chaotic behavior for certain parameters of nonintegrable case.

*Key words:* Discrete Solitons, Hirota Method, Singularity Confinement

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## 1 Introduction

Discrete solitons in nonlinear lattices have been the focus of considerable attention in diverse branches of science [1,2,3]. Discrete solitons are possible in several physical settings, such as biological systems [4], atomic chains [5,6],

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solid state physics [7], electrical lattices [8] and Bose-Einstein condensates [9]. Recently, the existence of discrete solitons in photonic structures (in arrays of coupled nonlinear optical waveguides [10,11,12,13,14,15,16] and in a nonlinear photonic crystal structure [17]) was announced and has attracted considerable attention in the scientific community. Photonic crystals, which are artificial microstructures having photonic bandgaps, can be used to precisely control propagation of optical pulses and beams. They are very useful for optical components such as waveguides, couplers, cavities and optical computers. It is possible to make discrete waveguides using photonic crystals. In this situation, “discrete solitons” appear naturally and have interesting properties. Many scientists believe that the discrete solitons can have an important role in this technology.

Here, we consider the physical system described by the discrete nonlinear Schrödinger (DNLS) equation. Most relevant for realistic applications (particularly to nonlinear optics) is the DNLS equation with a cubic (Kerr) nonlinearity:

$$i\frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n = 0, \quad (1)$$

where  $\psi_n$  are complex variables defined for all integer values of the site index  $n$ . The DNLS equation was used by Christodoulides and Joseph [11] to model the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides. In such an array, when low intensity light is injected into one, it will couple to more and more waveguides as it propagates, thereby broadening its spatial distribution (diffraction). High intensity light changes the refractive index of the input waveguides through the Kerr effect and decouples them from the rest of the array. Certain light distributions propagate while retaining a fixed spatial profile among a limited number of waveguides. These are discrete spatial solitons. Experimental results for optical waveguide arrays, confirming the validity of the model, have been reported by Eisenberg et al. [12].

There are many works about the DNLS equation and some its modifications. It is well-known that the standard DNLS equation is not completely integrable [18]. The integrable discrete nonlinear Schrödinger equation (Ablowitz-Ladik (AL) system),

$$i\frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0, \quad (2)$$

was proposed by using inverse scattering method [19]. The AL system has  $N$ -soliton solutions and a rich mathematical structure. However, it is not physically realistic, because it does not contain the Kerr-nonlinearity term. Thus

the deformed DNLS equation

$$i\frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + F(\psi_{n+1}, \psi_n, \psi_{n-1}) = 0, \quad (3)$$

where  $F(x, y, z)$  is a polynomial function of  $x, y, z$ , was proposed in several works. For example,

$$F(\psi_{n+1}, \psi_n, \psi_{n-1}) \equiv \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \quad (4)$$

was considered by several authors [20,21,22,23,?]. Equation (3) with Eq.(4) can be viewed as a deformation of the DNLS equation (1). Indeed, by using the parameterization  $\gamma = (\beta' - \beta)/2$ , one can see that, for  $\beta = \beta'$ , Eq.(3) with Eq.(4) reduces to the standard DNLS equation (1), while for  $\beta \rightarrow 0$  it gives the AL system. Kivshar and Salerno explained the physical meaning of Eq.(3) with Eq.(4) [21]. They proposed that an application of Eq.(3) with Eq.(4) may be found in nonlinear optics in, that the DNLS equation describes interactions of partial TE (Transverse electric) modes in array of (focusing or defocusing) waveguides. Thus, in spite of the fact that the AL system itself is not physical, it can appear as a particular case of a more general and physically better justified discrete model Eq.(3) with (4).

The existence of localized solutions is well known for spatially discrete systems (including quintic and higher order nonlinearities) [25,26,27,28]. Moreover the stability of these solutions at weak coupling is established as well, independent of the integrability of the underlying equations of motion.

However, nobody has found exact soliton-like solutions of Eq.(3)-type discrete systems. Concrete forms of localized solutions are, definitely, useful for analyzing real physics. Moreover, from physical point of view, investigating what kind of Eq.(3) has exact localized solutions and what kind of solution exits are important. Thus the problems we want to address here are: Does Eq.(3) with Eq.(4) have soliton-like solutions and what kind of deformed DNLS permits soliton-like solutions?

Here, we propose a new model of the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides, the quintic discrete nonlinear Schrödinger (QDNLS) equation

$$i\frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) + \delta|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = 0. \quad (5)$$

In physical problems, the quintic nonlinearity can be of equal or even higher importance to the cubic one [29] as it is responsible for stability of localized

solutions. This equation (in normalized coefficients  $\alpha = 1, \gamma = -4\delta/\beta$ ) can be derived from the Hamiltonian

$$H = - \sum_n (\psi_n \psi_{n+1}^* + \psi_n^* \psi_{n+1}) - \frac{\beta}{2\delta} \sum_n \ln(1 + \gamma |\psi_n|^2 + \delta |\psi_n|^4), \quad (6)$$

with the deformed Poisson brackets

$$\{\psi_n, \psi_m^*\} = i(1 + \gamma |\psi_n|^2 + \delta |\psi_n|^4) \delta_{nm}, \quad \{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0. \quad (7)$$

In general,

$$\{B, C\} = i \sum_n \left( \frac{\partial B}{\partial \psi_n} \frac{\partial C}{\partial \psi_n^*} - \frac{\partial C}{\partial \psi_n} \frac{\partial B}{\partial \psi_n^*} \right) (1 + \gamma |\psi_n|^2 + \delta |\psi_n|^4). \quad (8)$$

The equation of motion is

$$\dot{\psi}_n = \{H, \psi_n\}. \quad (9)$$

Surprisingly, the QDNLS equation has exact soliton solutions (bright and dark) and Jacobi elliptic function solutions, as we shall show here.

In this letter, we discuss the integrability of mappings which are reduced from the QDNLS equation and exact localized solutions of the QDNLS equation.

## 2 Integrability of mappings and exact localized solutions of the QDNLS equation

We begin by looking for solutions of the form  $\psi_n(t) = \phi_n e^{-i\omega t}$  where  $\phi_n$  is real. Substitution into Eq.(1), (4) and (5) yields the corresponding symmetric difference equations, respectively

$$\phi_{n+1} + \phi_{n-1} = (-\omega/\alpha + 2)\phi_n - (\beta/\alpha)\phi_n^3, \quad \alpha, \beta \neq 0, \quad (10)$$

$$\phi_{n+1} + \phi_{n-1} = \frac{(-\omega + 2\alpha)\phi_n - \beta\phi_n^3}{\alpha + \gamma\phi_n^2}, \quad \alpha, \beta, \gamma \neq 0, \quad (11)$$

and

$$\phi_{n+1} + \phi_{n-1} = \frac{(-\omega + 2\alpha)\phi_n - \beta\phi_n^3}{\alpha + \gamma\phi_n^2 + \delta\phi_n^4}, \quad \alpha, \beta, \delta \neq 0. \quad (12)$$

Equation (10) was studied by many authors [18] and has chaotic property. We apply a singularity confinement (SC) test to these difference equations [30]. Equation (11) does not pass the SC test except the case in which Eq.(11) is reduced to a linear mapping  $\phi_{n+1} + \phi_{n-1} = (-\beta/\gamma)\phi_n$  (in the case of  $(\omega - 2\alpha)\gamma = \alpha\beta$ ). Thus we may conclude that Eq.(11) is non-integrable except the case in which leads to a linear mapping. In the case in which Eq.(11) leads to a linear mapping, it has solutions  $\phi_n = \frac{1}{\sqrt{C^2-4}} \left\{ \left( \frac{C+\sqrt{C^2-4}}{C} \right)^n \left( \phi_1 - \frac{C-\sqrt{C^2-4}}{2} \phi_0 \right) - \left( \frac{C-\sqrt{C^2-4}}{C} \right)^n \left( \phi_1 - \frac{C+\sqrt{C^2-4}}{2} \phi_0 \right) \right\}$  where  $C = -\beta/\gamma$  and  $\phi_0, \phi_1$  are initial values.

In Eq.(12), divergences caused by initial values are confined to only one-site without losing the initial information for certain sets of coefficients. That is, under that constraint, Eq.(12) has the SC property. So Eq.(12) may be integrable for certain sets of coefficients. Indeed, this equation has soliton type solutions for certain sets of parameters.

Substituting  $\psi_n = \frac{g_n}{f_n} e^{-i\omega t}$  to Eq.(5), we obtain the multi-linear form

$$(\omega - 2\alpha)f_{n+1}f_n^3f_{n-1}g_n + \alpha f_n^4f_{n-1}g_{n+1} + \alpha f_{n+1}f_n^4g_{n-1} + \beta f_{n+1}f_n f_{n-1}g_n^3 \\ + \gamma f_n^2f_{n-1}g_{n+1}g_n^2 + \gamma f_{n+1}f_n^2g_n^2g_{n-1} + \delta f_{n-1}g_{n+1}g_n^4 + \delta f_{n+1}g_n^4g_{n-1} = 0.$$

Using the standard procedure of the Hirota method, we obtain the following exact solutions.

### Bright type soliton

$$\psi_n(t) = A \operatorname{sech}(n \log(p) + \log(n_0)) e^{-i\omega t}, \quad (13)$$

where  $\omega = \frac{\pm\beta\sqrt{\gamma^2-4\alpha\delta+4\alpha\delta+\beta\gamma}}{2\delta}$ ,  $p = \frac{\pm\sqrt{\omega(\omega-4\alpha)+2\alpha-\omega}}{2\alpha}$ ,  $A = \pm\frac{1}{2}\sqrt{\frac{\beta\omega(4\alpha-\omega)}{\alpha\delta(2\alpha-\omega)}}$  and  $n_0$  is arbitrary constant.

### Dark type soliton

$$\psi_n(t) = A \tanh(n \log(p) + \log(n_0)) e^{-i\omega t}, \quad (14)$$

where  $\omega = \frac{\pm\beta\sqrt{\gamma^2-4\alpha\delta+4\alpha\delta+\beta\gamma}}{2\delta}$ ,  $p = \sqrt{\frac{\pm\sqrt{2\alpha\omega+2\alpha+\omega}}{2\alpha-\omega}}$ ,  $A = \pm\sqrt{\frac{\beta\omega}{2\delta(2\alpha-\omega)}}$  and  $n_0$  is arbitrary constant.

The QDNLS equation also has Jacobi elliptic function solutions for certain sets of parameters.

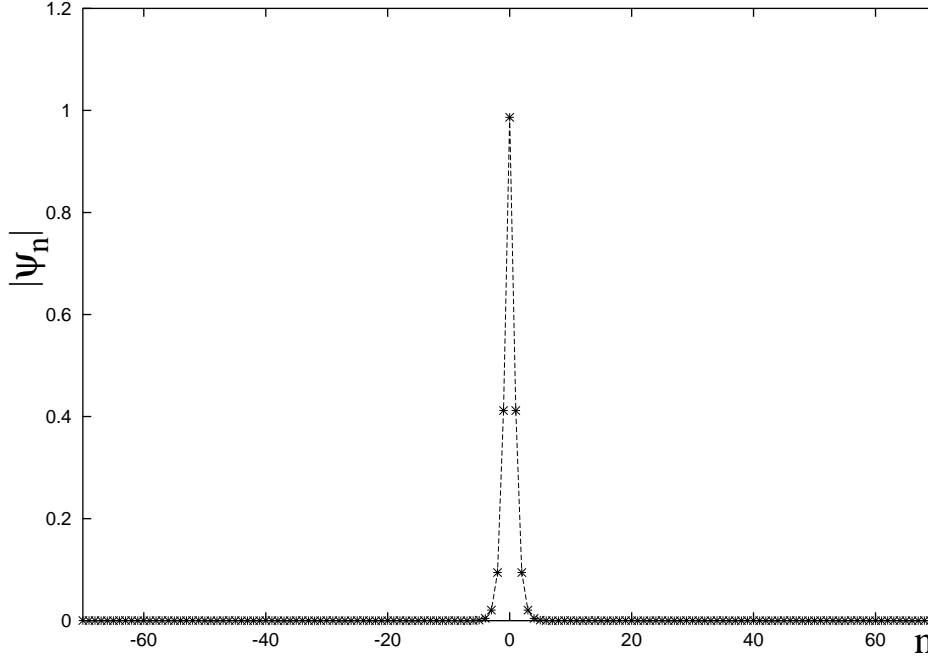


Fig. 1. Bright soliton of Eq.(5).  $\alpha = 1, \beta = 0.2, \gamma = 4.78701, \delta = -0.2$ . Time in this snapshot is 100.

Here, we show Fig.1, which is the result of numerical evolution of an initial exact bright soliton. We should note that the above localized solution is stable for certain parameters.

What happens for the sets of parameters for which Eq.(12) does not pass the SC test? We show two figures (  $(\phi_{n+1}, \phi_n)$  phase-plane plot) here. As can be seen, Fig.3 uses a set of non-integrable parameters and chaotic behavior is obtained (compare to Fig.2).

Let us explain why the mapping (12) is integrable for certain sets of parameters. It is well-known that the McMillan mapping [31]  $z_{n+1} + z_{n-1} = (A + Bz_n)/(C + Dz_n^2)$  has a conserved quantity and could be solved by Jacobi elliptic functions [32]. Equation (12) for certain sets of parameters reduces to the McMillan mapping by cancellation of common factors in denominator and numerator of Eq.(12). These cases of certain sets of parameters are equivalent to the cases in which Eq.(12) passes the SC test. For example, in the set of parameters of Fig.2, Eq.(12) reduces to the McMillan mapping  $\phi_{n+1} + \phi_{n-1} = -7\phi_n/(-22 + 11\phi_n^2)$ , which has an invariant  $-22(\phi_n^2 + \phi_{n-1}^2) + 7\phi_n\phi_{n-1} + 11\phi_n^2\phi_{n-1}^2$  and Jacobi elliptic function solutions [32].

We note that the quintic NLS equation with fourth order dispersion  $iu_t + au_{xx} + \epsilon bu_{xxx} + c|u|^2u + d|u|^4u = 0$  where  $\epsilon$  is a small parameter, corresponds to a continuous analogue of Eq.(5) [33,34,35]. This equation has exact bright (i.e.,

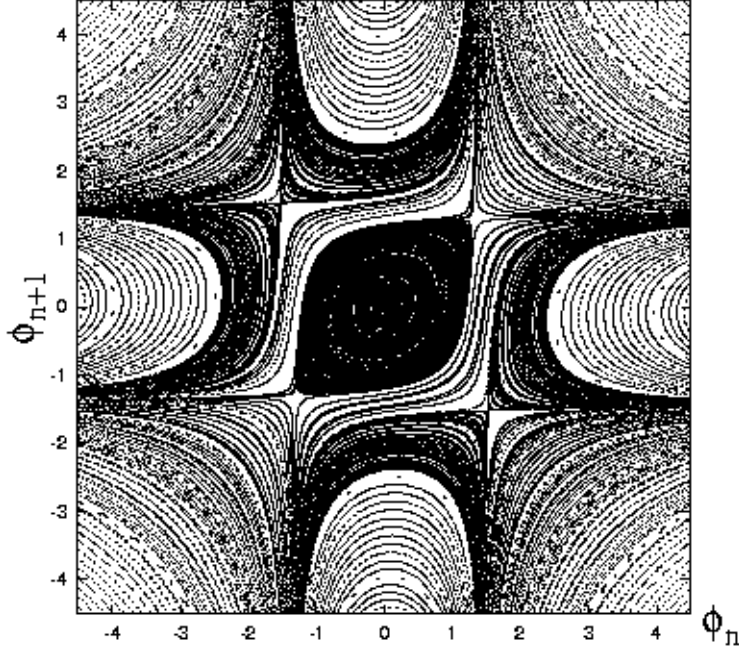


Fig. 2. A collection of orbits of the map generated by Eq. (12). (Integrable case:  $\omega = 37, \alpha = 22, \beta = 7, \gamma = -33, \delta = 11$ . These parameters were determined by the SC test. Each orbits are described by Jacobi elliptic functions.)

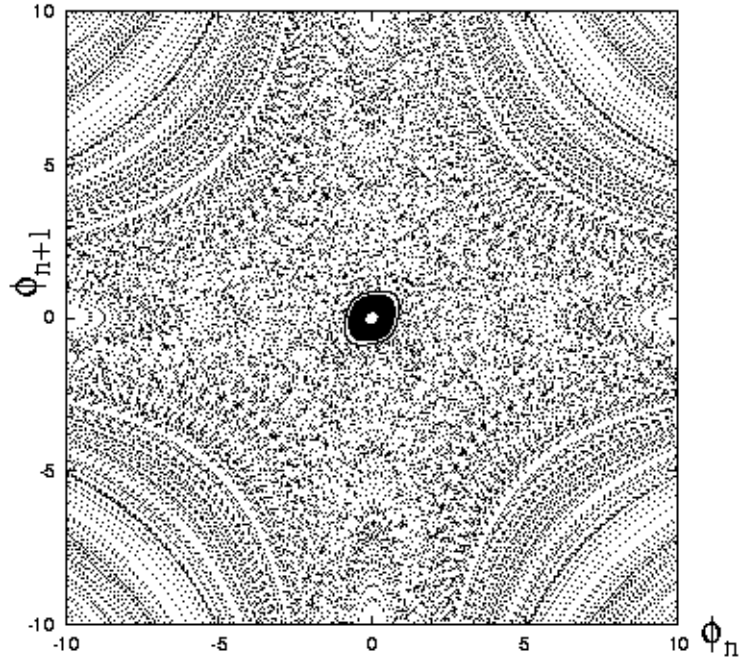


Fig. 3. A collection of orbits of the map generated by Eq.(12). (Non-integrable case:  $\omega = 37, \alpha = 22, \beta = 6, \gamma = -33, \delta = 11$ .)

$A \operatorname{sech}(kx)e^{it}$  ) and dark (i.e.,  $A \tanh(kx)e^{it}$  ) soliton solutions for certain sets of parameters [34,35,36]. The existence of exact localized solutions originates from the balance between quintic nonlinear term and 4th-order derivative term. This equation appears in the optical fiber transmission studies [34,35,37].

We can easily generalize to the following DNLS equation:

$$\begin{aligned}
i \frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \\
+ \gamma_2 |\psi_n|^2 (\psi_{n+1} + \psi_{n-1}) + \beta_2 |\psi_n|^2 \psi_n \\
+ \gamma_4 |\psi_n|^4 (\psi_{n+1} + \psi_{n-1}) + \beta_4 |\psi_n|^4 \psi_n \\
+ \dots \\
+ \gamma_{2N-2} |\psi_n|^{2N-2} (\psi_{n+1} + \psi_{n-1}) + \beta_{2N-2} |\psi_n|^{2N-2} \psi_n \\
+ \gamma_{2N} |\psi_n|^{2N} (\psi_{n+1} + \psi_{n-1}) = 0.
\end{aligned} \tag{15}$$

Using a solution ansatz of the form  $\psi_n(t) = \phi_n e^{-i\omega t}$ , where  $\phi_n$  is real, substitution into equations (15) yields the corresponding symmetric difference equation

$$\phi_{n+1} + \phi_{n-1} = \frac{(-\omega + 2\alpha)\phi_n - \beta_2 \phi_n^3 - \beta_4 \phi_n^5 - \dots - \beta_{2N-2} \phi_n^{2N-1}}{\alpha + \gamma_2 \phi_n^2 + \gamma_4 \phi_n^4 + \dots + \gamma_{2N-2} \phi_n^{2N-2} + \gamma_{2N} \phi_n^{2N}}. \tag{16}$$

This mapping has the SC property and low-order degree growth property when the coefficients satisfy some constraints. In fact, it is also possible to construct some exact solutions by using standard Hirota method. We may consider the existence of these exact localized solutions also originates from the balance between  $2N+1$  order nonlinear term and  $2N$ -order derivative term which comes from the higher order term of Taylor expansion of second-order difference term.

### 3 Non-integrable mappings and their growth properties

Next, we consider the forced quintic discrete Nonlinear Schrödinger (fQDNLS) equation

$$\begin{aligned}
i \frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta |\psi_n|^2 \psi_n + \gamma |\psi_n|^2 (\psi_{n+1} + \psi_{n-1}) \\
+ \delta |\psi_n|^4 (\psi_{n+1} + \psi_{n-1}) + f \exp(-i\omega t) = 0,
\end{aligned} \tag{17}$$



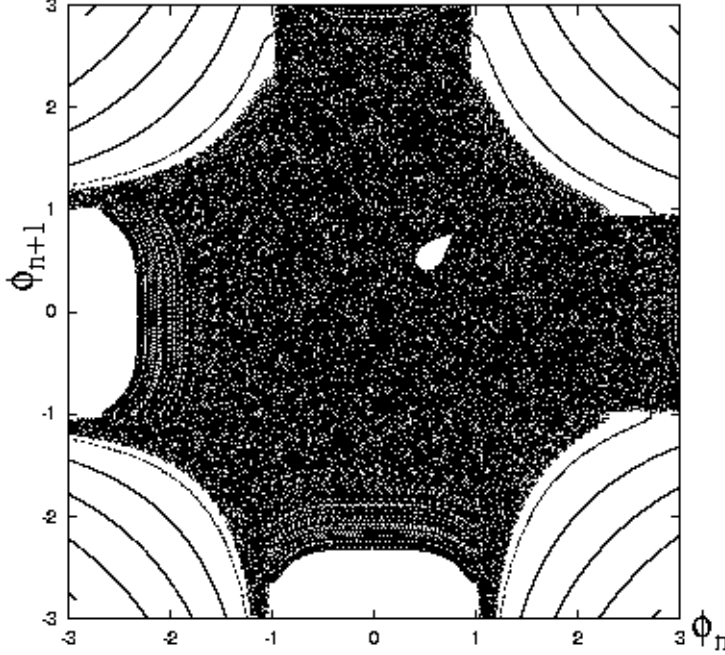


Fig. 4. A collection of orbits of the map generated by Eq.(19).

where  $f$  is a constant parameter. This equation can be reduced by  $\psi_n = \phi_n \exp(-i\omega t)$  into

$$\phi_{n+1} + \phi_{n-1} = \frac{f + (-\omega + 2\alpha)\phi_n - \beta\phi_n^3}{\alpha + \gamma\phi_n^2 + \delta\phi_n^4}. \quad (18)$$

Our problem is whether this is integrable or not. After we apply the SC test, we note Eq.(18), for certain sets of parameters passes the SC test without losing initial information. For example, we can consider the simple mapping

$$\phi_{n+1} + \phi_{n-1} = 1/(1 - \phi_n^4). \quad (19)$$

This mapping has the SC property. But this result is a counterexample to the physical intuition that nonlinear equations containing forcing terms are chaotic. To verify this, we perform numerical calculation on Eq.(19). The result obtained from Eq.(19) is shown respectively as  $(\phi_{n+1}, \phi_n)$  phase-plane plots in Fig.4 for varying choices of  $\phi_0, \phi_1$ . (In contrast,  $(\phi_{n+1}, \phi_n)$  phase-plane plots (Fig.2) of Eq.(12) show periodic orbits.)

This situation is similar to the case of Hietarinta and Viallet [38]. Algebraic entropy is useful to check integrability in this case [39,40,41]. The main argument is that a generic, non-integrable mapping has an exponential degree growth, while integrability is associated with low growth and is typically polynomial. The degrees of the iterates of Eq.(19) are 0,1,4,17,64,241,900,..., i.e. a

exponential degree growth. Thus we can assert that Eq.(19) is non-integrable. This fact can be confirmed by considering cancellation mechanism of common factors in denominator and numerator of Eq.(19) (or Eq.(18)). The denominator and numerator of Eq.(19) (or Eq.(18)) do not have common factors. Thus Eq.(19) (or Eq.(18)) cannot be reduced to the McMillan mapping.

From above analysis, we can easily produce the following generalized mapping:

$$\phi_{n+1} + \phi_{n-1} = \frac{a_0 + a_1\phi_n + \cdots + a_{N-1}\phi_n^{N-1}}{b_0 + b_1\phi_n + b_2\phi_n^2 + \cdots + b_N\phi_n^N}. \quad (20)$$

This mapping also passes the SC test for certain sets of parameters, but has exponential degree growth. Thus  $(\phi_{n+1}, \phi_n)$  phase-plane plots of Eq.(20) also trace chaotic orbits.

## 4 Conclusions

We have analyzed the QDNLS equation. The QDNLS equation is not completely integrable, however the QDNLS equation has a number of exact localized solutions exists to the QDNLS equation provided that coefficients are bound by special relations (i.e. partially integrable). The set of localized solutions include particular types of solitary wave solutions, dark soliton solutions and periodic solutions in terms of Jacobi elliptic functions. Clearly these localized solutions represent only a small subset of large variety of possible solutions admitted by the QDNLS equation. Nevertheless, the solutions presented here are found for the first time and they might serve as seeding solutions for a wider class of localised structures which exist in this system. We hope that they will be useful in further perturbative and numerical analysis of various solutions to the QDNLS equation.

We mention traveling wave solutions of the QDNLS equation. Flach et al. studied moving pulses of the generalized discrete nonlinear Schrödinger equation[42]

$$i\frac{d\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + F(|\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) + G(|\psi_n|^2)\psi_n = 0. \quad (21)$$

Their result shows that the AL system is only generalized discrete nonlinear Schrödinger equation with sech-type moving pulses, i.e. the QDNLS equation does not have exact sech-type exact traveling wave solutions. Although the QDNLS equation does not have exact traveling wave solutions, there is still the

possibility that non-exact traveling wave solutions exist. Such solutions must conserve the integrals of motion (number of soliton, energy and momentum).

We have also found interesting properties of mappings derived from the deformed DNLS equation.

Further studies (e.g. stability of exact localized solutions and bound states of multiple pulses, and the existence and properties of travelling wave solutions) will be addressed elsewhere. We believe our result can be applied to the study of discrete optical (and other) solitons.

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